Local $U(1)$ symmetry in $Y(SO(5))$ associated with the massless Thirring model and its Bethe ansatz

2000 J. Phys. A: Math. Gen. 33 L345

(http://iopscience.iop.org/0305-4470/33/37/101)

J. Phys. A: Math. Gen. **33** (2000) L345–L352. Printed in the UK PII: S0305-4470(00)14052-1

LETTER TO THE EDITOR

Local $U(1)$ symmetry in $Y(SO(5))$ associated with the massless Thirring model and its Bethe ansatz

Hong-Biao Zhang†‡, Mo-Lin Ge† and Kang Xue†§

† Theoretical Physics Division, Nankai Institute of Mathematics, Nankai University, Tianjin 300071, People's Republic of China

‡ Educational Institute of Jilin Province, Changchun, Jilin 130022, People's Republic of China

§ Physics Department, Northeast Normal University, Changchun, Jilin 130024, People's Republic of China

Received 18 May 2000

Abstract. The massless Thirring model associated with $SO(5)$ is solved in terms of the local $U(1)$ symmetry. The local $U(1)$ symmetry is related to q-deformation of four-component field operators due to the nonlinear interaction for different internal degrees of freedom. The Bethe ansatz wavefunction is also discussed. In addition, the local $U(1)$ symmetry in the Yangian associated with $SO(5)(Y(SO(5)))$ is explored.

1. Introduction

Recently, it has been proposed by Zhang *et al* that the antiferromagnetic (AF) and superconducting (SC) phases of high-T_c cuprates are unified by an approximated $SO(5)$ symmetry principle [1]. Considerable support for this proposal came from numerical investigations in models for high-T_c materials. In particular, it was shown that the low-energy excitations can be classified in terms of an $SO(5)$ symmetry multiplet structure [2, 3]. Subsequently, extended Hubbard models and a two-leg ladder model related to $SO(5)$ symmetry have been introduced and analysed in detail [4–6]. On the other hand, Shelton and Sénéchal [7] have studied the problem of two coupled 1D Tomonaga–Luttinger chains and concluded that approximate $SO(5)$ symmetry can emerge in the low-energy limit of this model. It is well known that the Luttinger liquid is connected with the massless Thirring model. It is worthwhile to deal with the massless Thirring model with $SO(5)$ symmetry. The model can be constructed by the four-component fermionic field operator $\psi_i(x)$; we shall show that this model is exactly solvable by the Bethe ansatz method through a local $U(1)$ transformation, under which the fermionic operator $\psi_i(x)$ is transformed into a q-deformed fermionic operator $\Phi_i(x)$. This procedure leads to the diagonalization that is shown in a simple manner by Wadati [8–10]. Furthermore, the study of Yangian algebra [11–14] provides a significant tool in the formalism of integrable models. The generators of the Yangian can be realized through currents for a given Lie algebra. It turns out that the current realization of $Y(SO(5))$ is not unique and allows a local $U(1)$ gauge transformation. It is interesting to find the consequence of such a $U(1)$-freedom according to the q-deformation of the fermionic operator $\Phi_i(x)$.

This paper is organized as follows. In section 2, the massless Thirring model with $SO(5)$ symmetry will be diagonalized and the Bethe ansatz wavefunction constructed. In section 3, we shall give the current algebra realization of $Y(SO(5))$ in terms of q-deformed fermionic current that gives rise to the local $U(1)$-gauge transformation.

2. The massless Thirring model with $SO(5)$ symmetry and its Bethe ansatz wavefuction

Let us consider the massless Thirring model constructed by the four-component fermion field operator $\psi(x) = [\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)]^T$. The Hamiltonian takes the form

$$H = \int \left[\mathrm{i}v \sum_{i=1}^{4} C_i \psi_i^+(x) \partial_x \psi_i(x) + g \sum_{i,j=1}^{4} C_{ij} n_i(x) n_j(x) \right] \mathrm{d}x \tag{1}$$

where $C_{ij} = C_{ji}, C_{ii} = 0$ and $n_i(x) = \psi_i^+(x)\psi_i(x)$ $(i, j = 1, 2, 3, 4)$ satisfy the anticommutation relations

$$[\psi_i^+(x), \psi_j^+(y)]_+ = 0 \tag{2}$$

$$[\psi_i(x), \psi_j(y)]_+ = 0 \tag{3}$$

$$[\psi_i(x), \psi_j^+(y)]_+ = \delta_{ij}\delta(x - y). \tag{4}$$

For the four-component fermionic field operator $\psi(x) = [c_\sigma(x), d_\sigma^+(x)]^T$ and forms the current algebra obeying $SO(5)$ [6]. In momentum space, this Hamiltonian can be written as

$$\begin{aligned} H = \int \bigg[-v \sum_{i=1}^{4} k C_i n_i(k) \bigg] \mathrm{d}k \\ &+ \frac{g}{\pi} \iiint \bigg[C_{12} c_\uparrow^+\Big(k + \frac{q}{2}\Big) c_\downarrow^+\Big(-k + \frac{q}{2}\Big) c_\downarrow\Big(-k' + \frac{q}{2}\Big) c_\uparrow\Big(k' + \frac{q}{2}\Big) \\ &+ C_{13} c_\uparrow^+\Big(k + \frac{q}{2}\Big) d_\uparrow^+\Big(-k + \frac{q}{2}\Big) d_\uparrow\Big(-k' + \frac{q}{2}\Big) c_\uparrow\Big(k' + \frac{q}{2}\Big) \\ &+ C_{14} c_\uparrow^+\Big(k + \frac{q}{2}\Big) d_\downarrow^+\Big(-k + \frac{q}{2}\Big) d_\downarrow\Big(-k' + \frac{q}{2}\Big) c_\uparrow\Big(k' + \frac{q}{2}\Big) \\ &+ C_{23} c_\downarrow^+\Big(k + \frac{q}{2}\Big) d_\uparrow^+\Big(-k + \frac{q}{2}\Big) d_\uparrow\Big(-k' + \frac{q}{2}\Big) c_\downarrow\Big(k' + \frac{q}{2}\Big) \\ &+ C_{24} c_\downarrow^+\Big(k + \frac{q}{2}\Big) d_\downarrow^+\Big(-k + \frac{q}{2}\Big) d_\downarrow\Big(-k' + \frac{q}{2}\Big) c_\downarrow\Big(k' + \frac{q}{2}\Big) \\ &+ C_{34} d_\uparrow^+\Big(k + \frac{q}{2}\Big) d_\downarrow^+\Big(-k + \frac{q}{2}\Big) d_\downarrow\Big(-k' + \frac{q}{2}\Big) d_\uparrow\Big(k' + \frac{q}{2}\Big) \bigg] \mathrm{d}k\, \mathrm{d}k'\, \mathrm{d}q \end{aligned} \tag{5}$$

that obviously is made up of pairs, so this model may be applied to SC.

To diagonalize H, we introduce the local $U(1)$ transformation

$$\Phi_i(x) = \exp\bigg[-\mathrm{i} \sum_{k=1}^{4} \theta_{ik} \phi_k(x) \bigg] \psi_i(x) \tag{6}$$

where $\phi_i(x) = \int_{-\infty}^{x} \psi_i^+(y)\psi_i(y)\,\mathrm{d}y$ and θ_{ik} are constants.

According to equations (2)–(4) and (6) by direct calculation, we obtain (no summation over the repeated j)

$$\Phi_i(x)\Phi_j(y) = -\exp[\mathrm{i}\theta_{ij}]\Phi_j(y)\Phi_i(x) \tag{7}$$

$$\Phi_i^+(x)\Phi_j^+(y) = -\exp[\mathrm{i}\theta_{ij}]\Phi_j^+(y)\Phi_i^+(x) \tag{8}$$

$$\Phi_i(x)\Phi_j^+(y) = -\exp[-\mathrm{i}\theta_{ij}]\Phi_j^+(y)\Phi_i(x) + \delta_{ij}\delta(x - y). \tag{9}$$

This is a special case of Zamolodchikov–Faddeev algebra [15, 16]:

$$\theta_{ii} = 0 \qquad (\mathrm{mod}\ 2\pi) \tag{10}$$

$$\theta_{ij} + \theta_{ji} = 0 \qquad (\mathrm{mod}\ 2\pi). \tag{11}$$

Therefore, equations (10) and (11) are conditions given by the associativity of the special case of Zamolodchikov–Faddeev algebra. The meaning of equation (10) is clear: that the particle itself must still be a fermion for the same 'i-spin' states; however, equation (11) show that the commutation relations between different 'i-spin' states can be q-deformed and the q-deformation parameters should obey equation (11) because of the two-body interaction between different 'i-spin' states.

Under the local $U(1)$ transformation equation (6), the Hamiltonian equation (1) can be diagonalized and we can find the physical constraint conditions for real C_i and C_{ij}. The Heisenberg equation $\mathrm{i}\partial_t \psi_i(x,t) = [\psi_i(x,t), H]$ reads

$$\partial_t \psi_i(x,t) = vC_i \partial_x \psi_i(x,t) - \mathrm{i}2g \sum_{j=1}^{4} C_{ij} n_j(x,t) \psi_i(x,t). \tag{12}$$

On account of the transformation equation (6) and the Heisenberg equation (12), we obtain

$$\partial_t \Phi_i(x,t) - vC_i \partial_x \Phi_i(x,t) = \mathrm{i} \sum_{j=1}^{4} [v(C_i - C_j)\theta_{ij} - 2gC_{ij}] n_j(x,t) \psi_i(x,t) \times \exp\left[-\mathrm{i} \sum_{k=1}^{4} \theta_{jk} \phi_k(x) \right]. \tag{13}$$

By choosing

$$\theta_{ij} = \frac{2g}{v} \frac{C_{ij}}{C_i - C_j} \qquad (C_i \neq C_j) \tag{14}$$

$$\theta_{ii} = 0 \tag{15}$$

$\Phi_i(x,t)$ satisfy the free-field equation. The Hamiltonian becomes diagonalized (here we suppose $C_i \neq C_j$; if $C_i = C_j$, the Hamiltonian can be diagonalized only when $C_{ij} = 0$):

$$H' = \mathrm{i}v \sum_{i=1}^{4} C_i \int \Phi_i^+(x) \partial_x \Phi_i(x)\, \mathrm{d}x. \tag{16}$$

The direct calculation shows that $\Phi_i(x)$ and H' also satisfy the Heisenberg equation $\mathrm{i}\partial_t \Phi_i(x,t) = [\Phi_i(x,t), H']$, so $\Phi_i(x,t)$ are really dynamic variables regarding H'.

Therefore, by using the local $U(1)$ transformation equation (6), the original Hamiltonian equation (1) constructed by $\psi_i(x)$ with anticommutation relation equations (2)–(4) has been transformed into the quadratic Hamiltonian equation (16) in terms of the $\Phi_i(x)$ obeying q-deformed relation equations (7)–(9). In the following, we shall show how the method in [8–10] works to find the Bethe ansatz wavefunction in a simple manner for the $SO(5)$ massless Thirring model.

Let us denote by $| n_1, n_2, n_3, n_4\rangle$ an eigenstate with n_i Φ_i-particles ($i = 1, 2, 3, 4$); it can be expressed by

$$|n_1, n_2, n_3, n_4\rangle = \int \cdots \int \prod_{j=1}^{M} \mathrm{d}x_j\, \varphi(x_1, \ldots, x_M) \prod_{j_1=1}^{n_1} \Phi_1^+(x_{j_1}) \times \prod_{j_2=1}^{n_2} \Phi_2^+(x_{M_1+j_2}) \prod_{j_3=1}^{n_3} \Phi_3^+(x_{M_2+j_3}) \prod_{j_4=1}^{n_4} \Phi_4^+(x_{M_3+j_4}) |0\rangle \tag{17}$$

where $M_i = n_1 + n_2 + \ldots + n_i$, $M = M_4$ and $|0\rangle$ is the vacuum defined by

$$\psi_j(x)|0\rangle = 0 \tag{18}$$

or equivalently

$$\Phi_j(x)|0\rangle = 0. \tag{19}$$

Substituting equations (16) and (17) into the Schrödinger equation

$$H'|n_1, n_2, n_3, n_4\rangle = E_{n_1,n_2,n_3,n_4}|n_1, n_2, n_3, n_4\rangle \tag{20}$$

yields an equation for $\varphi(x_1, \ldots, x_M)$:

$$\mathrm{i}v\left(\sum_{i=1}^{4} C_i \sum_{j_i=1}^{n_i} \frac{\partial}{\partial x_{M_{i-1}+j_i}}\right)\varphi(x_1, \ldots, x_M) = E_{n_1,n_2,n_3,n_4}\varphi(x_1, \ldots, x_M) \tag{21}$$

whose solution is

$$\begin{aligned} \varphi(x_1, \ldots, x_M) &= A \exp\left(\mathrm{i}\sum_{j=1}^{M} k_j x_j\right) \\ E_{n_1,n_2,n_3,n_4} &= -v\left(\sum_{i=1}^{4} C_i \sum_{j_i=1}^{n_i} k_{M_{i-1}+j_i}\right) \end{aligned} \tag{22}$$

where k_j and A are constants. Since the constant A is not essential, we shall omit it hereafter. The Bethe ansatz wavefunction $\hat{\varphi}(x_1, \ldots, x_M)$ is defined by

$$\begin{aligned} |n_1, n_2, n_3, n_4\rangle = \int \cdots \int \prod_{j=1}^{M} \mathrm{d}x_j \hat{\varphi}(x_1, \ldots, x_M) \prod_{j_1=1}^{n_1} \psi_1^+(x_{j_1}) \\ \times \prod_{j_2=1}^{n_2} \psi_2^+(x_{M_1+j_2}) \prod_{j_3=1}^{n_3} \psi_3^+(x_{M_2+j_3}) \prod_{j_4=1}^{n_4} \psi_4^+(x_{M_3+j_4})|0\rangle. \end{aligned} \tag{23}$$

Substituting equation (6) into (17), by detailed calculation, we have

$$\begin{aligned} |n_1, n_2, n_3, n_4\rangle = \int \cdots \int \prod_{j=1}^{M} \mathrm{d}x_j \varphi(x_1, \ldots, x_M) \prod_{1\leqslant p<q\leqslant 4} \prod_{j_p=1}^{n_p} \prod_{j_q=1}^{n_q} \\ \times \exp[\mathrm{i}\theta_{pq}\theta(x_{M_{p-1}+j_p} - x_{M_{q-1}+j_q})] \\ \times \prod_{j_1=1}^{n_1} \psi_1^+(x_{j_1}) \prod_{j_2=1}^{n_2} \psi_2^+(x_{M_1+j_2}) \prod_{j_3=1}^{n_3} \psi_3^+(x_{M_2+j_3}) \prod_{j_4=1}^{n_4} \psi_4^+(x_{M_3+j_4})|0\rangle \\ \sim \int \cdots \int \prod_{j=1}^{M} \mathrm{d}x_j \varphi(x_1, \ldots, x_M) \prod_{1\leqslant p<q\leqslant 4} \prod_{j_p=1}^{n_p} \prod_{j_q=1}^{n_q} \\ \times \left[1 - \mathrm{i}\tan\frac{\theta_{pq}}{2}\epsilon(x_{M_{p-1}+j_p} - x_{M_{q-1}+j_q})\right] \\ \times \prod_{j_1=1}^{n_1} \psi_1^+(x_{j_1}) \prod_{j_2=1}^{n_2} \psi_2^+(x_{M_1+j_2}) \prod_{j_3=1}^{n_3} \psi_3^+(x_{M_2+j_3}) \prod_{j_4=1}^{n_4} \psi_4^+(x_{M_3+j_4})|0\rangle \end{aligned} \tag{24}$$

where $\theta(x) = 0$ (if $x < 0$); 1 (if $x > 0$) and $\epsilon(x) = \theta(x) - \theta(-x)$ hereafter. Thus, the Bethe ansatz wavefunction $\hat{\varphi}(x_1, \ldots, x_M)$ takes the form

$$\hat{\varphi}(x_1, \ldots, x_M) = \exp\left[\mathrm{i}\sum_{j=1}^{M} k_j x_j\right] \prod_{1\leqslant p<q\leqslant 4} \prod_{j_p=1}^{n_p} \prod_{j_q=1}^{n_q} \left[1 - \mathrm{i}tg\frac{\theta_{pq}}{2}\epsilon(x_{M_{p-1}+j_p} - x_{M_{q-1}+j_q})\right] \tag{25}$$

which describes the many-body problem with δ-interactions.

Suppose that M particles move in a region with the length L. For an arbitrary $x_j(M_{p-1} \leqslant j \leqslant M_p)$, imposing the periodical boundary conditions (PBCs), we have

$$k_j L = -\mathrm{i} \sum_{\substack{q \neq p \\ q=1}}^{4} n_q \ln \frac{1 - \mathrm{i} tg\theta_{pq}/2}{1 + \mathrm{i} tg\theta_{pq}/2} + 2l_j\pi \qquad (l_j \text{ integer}) \tag{26}$$

i.e.

$$k_j L = -\sum_{\substack{q \neq p \\ q=1}}^{4} n_q \theta_{pq} + 2l_j\pi \qquad (l_j \text{ integer}) \tag{27}$$

that is exactly the Bethe ansatz equation. Obviously, the local $U(1)$ transformation equation (6) greatly helps the derivation of the Bethe ansatz condition for the massless Thirring model.

3. Current realization of $Y(SO(5))$

The $SO(5)$ algebra does have a current realization; however, the fermionic construction is not unique. In parallel to the diagonalization of equation (1) we shall show that the q-deformed operators $\Phi_i(x)$ shown in equation (6) also provide a realization of $SO(5)$ algebra, henceforth the Yangian associated with $SO(5)$.

The original commutation relations of $Y(g)$ were given by Drinfled [17, 18] in the form

$$[I_\lambda, I_\mu] = c_{\lambda\mu\nu} I_\nu \qquad [I_\lambda, J_\mu] = c_{\lambda\mu\nu} J_\nu \tag{28}$$

$$[J_\lambda, [J_\mu, I_\nu]] - [I_\lambda, [J_\mu, J_\nu]] = h^2 a_{\lambda\mu\nu\alpha\beta\gamma}\{I_\alpha, I_\beta, I_\gamma\} \tag{29}$$

$$[[J_\lambda, J_\mu], [I_\sigma, J_\tau]] + [[J_\sigma, J_\tau], [I_\lambda, J_\mu]] = h^2(a_{\lambda\mu\nu\alpha\beta\gamma} c_{\sigma\tau\nu} + a_{\sigma\tau\nu\alpha\beta\gamma} c_{\lambda\mu\nu})\{I_\alpha, I_\beta, I_\gamma\} \tag{30}$$

where $c_{\lambda\mu\nu}$ are structure constants of a simple Lie algebra g, h is a constant and

$$a_{\lambda\mu\nu\alpha\beta\gamma} = \frac{1}{4!} c_{\lambda\alpha\sigma} c_{\mu\beta\tau} c_{\nu\gamma\rho} c_{\sigma\tau\rho} \qquad \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k. \tag{31}$$

For Lie algebra $SO(5)$, $Y(SO(5))$ is generated by antisymmetric generators $\{I_{ab}, J_{ab}\}$. Equation (28) reads

$$[I_{ab}, I_{cd}] = \mathrm{i}(\delta_{bc} I_{ad} + \delta_{ad} I_{bc} - \delta_{ac} I_{bd} - \delta_{bd} I_{ac}) \tag{32}$$

$$\begin{aligned} &[I_{ab}, J_{cd}] = \mathrm{i}(\delta_{bc} J_{ad} + \delta_{ad} J_{bc} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac}) \\ &I_{ab} = -I_{ba} \qquad J_{ab} = -J_{ba} \qquad (a, b, c, d = 1, 2, 3, 4, 5). \end{aligned} \tag{33}$$

Not all of the relations in equations (29) and (30) are independent. After tedious calculation we can prove that there is only one independent relation:

$$[J_{23}, J_{15}] = \frac{\mathrm{i}}{24} h^2(\{I_{13}, I_{42}, I_{45}\} + \{I_{12}, I_{45}, I_{34}\} - \{I_{14}, I_{42}, I_{35}\} - \{I_{14}, I_{34}, I_{25}\}) \tag{34}$$

where J_{23} and J_{15} are the Cartan subset.

All the relations other than equation (28) can be generated on the basis of equation (34) by using Jacobi identities together with equations (32) and (33). Therefore, for $Y(SO(5))$, equations (28)–(30) can be expressed with equations (32)–(34) in such a simple manner. This conclusion can also be verified by the RTT relation independently through tremendous computation.

The generators of $Y(SO(5))$ can be realized by fermionic current algebra as follows:

$$\begin{aligned} I_{ab} &= \int I_{ab}(x)\,\mathrm{d}x \qquad & I_{ab}(x) &= -\tfrac{1}{2}\psi^{+}(x)\Gamma^{ab}\psi(x) \\ J_{ab} &= T_{ab} + UJ^{0}_{ab} \qquad & T_{ab} &= \int \mathrm{d}x\ \psi^{+}(x)\Gamma^{ab}\partial_x\psi(x) \\ J^{0}_{ab} &= \iint \mathrm{d}x\,\mathrm{d}y\,\epsilon(x-y)I_{ac}(x)I_{cb}(y) \end{aligned} \tag{35}$$

where $\Gamma^{ab} = -\mathrm{i}\Gamma^{a}\Gamma^{b}$, Γ^{a} are 4×4 Dirac matrices, $U = \pm\frac{\mathrm{i}}{2}h$ (h being an arbitrary constant) and $\psi(x)$ satisfies anticommutation relation equations (2)–(4). It can be checked that the set $\{I_{ab}, J_{ab}\}$ satisfies algebraic relation equations (32)–(34) of $Y(SO(5))$.

As given by [6] if $\psi(x) = [c_\sigma(x), d_\sigma(x)]^T$, then local generators $I_{ab}(x)$ of Lie algebra $SO(5)$ are expressed in terms of spin $\vec{S}(x) = \frac{1}{2}(c^{+}(x)\vec{\sigma}c(x) + d^{+}(x)\vec{\sigma},\ d(x))$, charge $Q(x) = \frac{1}{2}(c^{+}(x)c(x) + d^{+}(x)d(x) - 2)$ and $\vec{\pi}^{+}(x) = -\frac{1}{2}c^{+}(x)\vec{\sigma}\sigma_2 d^{+}(x)$ with

$$I_{ab}(x) = \begin{pmatrix} 0 & & & & \\ \pi_1^{+}(x)+\pi_1(x) & 0 & & & \\ \pi_2^{+}(x)+\pi_2(x) & -S_3(x) & 0 & & \\ \pi_3^{+}(x)+\pi_3(x) & S_2(x) & -S_1(x) & 0 & \\ Q(x) & \mathrm{i}(\pi_1(x)-\pi_1^{+}(x)) & \mathrm{i}(\pi_2(x)-\pi_2^{+}(x)) & \mathrm{i}(\pi_3(x)-\pi_3^{+}(x)) & 0 \end{pmatrix} \tag{36}$$

where the values of matrix elements on the upper right triangle are determined by antisymmetry, $I_{ab}(x) = -I_{ba}(x)$.

Under the local $U(1)$ transformation equation (6), the four-component fermionic field operator $\psi(x) = [\psi_1(x), \psi_2(x), \psi_3(x), \psi_4(x)]^T$ is changed into the q-deformed operator $\Phi(x) = [\Phi_1(x), \Phi_2(x), \Phi_3(x), \Phi_4(x)]^T$. The generators of $Y(SO(5))$ are constructed by q-deformed fermionic current algebra as follows:

$$\begin{aligned} \overline{I}_{ab} &= \int \overline{I}_{ab}(x)\,\mathrm{d}x \qquad & \overline{I}_{ab}(x) &= -\tfrac{1}{2}\Phi^{+}(x)\Gamma^{ab}\Phi(x) \\ \overline{J}_{ab} &= \overline{T}_{ab} + U\overline{J}^{0}_{ab} \qquad & \overline{T}_{ab} &= \int \mathrm{d}x\ \Phi^{+}(x)\Gamma^{ab}\partial_x\Phi(x) \\ \overline{J}^{0}_{ab} &= \iint \mathrm{d}x\,\mathrm{d}y\,\epsilon(x-y)\overline{I}_{ac}(x)\overline{I}_{cb}(y). \end{aligned} \tag{37}$$

Substituting equation (37) into (32)–(34), we can obtain the constraint conditions

$$\begin{aligned} &\theta_{im} - \theta_{jm} = \theta_{in} - \theta_{jn} \qquad (\mathrm{mod}\ 2\pi) \\ &(i, j, m, n = 1, 2, 3, 4) \end{aligned} \tag{38}$$

where equation (38) sets the condition making $Y(SO(5))$ constructed by the q-deformed field operator $\Phi(x)$. This indicates that the current realization of $Y(SO(5))$ is not unique. Careful calculation shows that there are three free parameters in θ_{ij} under condition equations (10), (11) and (38), so there exists an additional freedom in $Y(SO(5))$. Substituting equation (14) into (38), we obtain

$$\frac{C_{im}}{C_i - C_m} - \frac{C_{jm}}{C_j - C_m} = \frac{C_{in}}{C_i - C_n} - \frac{C_{jn}}{C_j - C_n} \qquad (\mathrm{mod}\ 2\pi) \quad (i, j, m, n = 1, 2, 3, 4). \tag{39}$$

In another words, H' is the Hamiltonian expressed by $\Phi_i(x)$; under the transformation equation (6), it becomes H where two-body interaction appears. Consequently the physical meaning of $U(1)$ transformation in $Y(SO(5))$ is connected with the two-body interaction in the massless Thirring model with $SO(5)$ symmetry.

From the above analysis we see that there is a local $U(1)$ gauge invariance in the construction of the current algebra realization for $Y(SO(5))$.

Under the local $U(1)$ transformation equations (6), (36) is changed into

$$\overline{I}_{ab}(x)=\begin{pmatrix} 0 & & & & \\ \overline{\pi}_1^+(x)+\overline{\pi}_1(x) & 0 & & & \\ \overline{\pi}_2^+(x)+\overline{\pi}_2(x) & -\overline{S}_3(x) & 0 & & \\ \overline{\pi}_3^+(x)+\overline{\pi}_3(x) & \overline{S}_2(x) & -\overline{S}_1(x) & 0 & \\ \overline{Q}(x) & \mathrm{i}(\overline{\pi}_1(x)-\overline{\pi}_1^+(x)) & \mathrm{i}(\overline{\pi}_2(x)-\overline{\pi}_2^+(x)) & \mathrm{i}(\overline{\pi}_3(x)-\overline{\pi}_3^+(x)) & 0 \end{pmatrix} \tag{40}$$

where the value of matrix elements on the upper right triangle are determined by antisymmetry, $\overline{I}_{ab}(x)=-\overline{I}_{ba}(x)$, and

$$\begin{pmatrix}\overline{S}_1(x)\\ \overline{S}_2(x)\\ \overline{S}_3(x)\end{pmatrix}=\begin{pmatrix} \cos\left(\frac{\alpha+\beta}{2}\phi(x)\right)\cos\left(\frac{\alpha-\beta}{2}\phi(x)\right) & -\sin\left(\frac{\alpha+\beta}{2}\phi(x)\right)\cos\left(\frac{\alpha-\beta}{2}\phi(x)\right) & 0\\ \sin\left(\frac{\alpha+\beta}{2}\phi(x)\right)\cos\left(\frac{\alpha-\beta}{2}\phi(x)\right) & \cos\left(\frac{\alpha+\beta}{2}\phi(x)\right)\cos\left(\frac{\alpha-\beta}{2}\phi(x)\right) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}S_1(x)\\ S_2(x)\\ S_3(x)\end{pmatrix}$$
$$+\begin{pmatrix} -\sin\left(\frac{\alpha+\beta}{2}\phi(x)\right)\sin\left(\frac{\alpha-\beta}{2}\phi(x)\right) & -\cos\left(\frac{\alpha+\beta}{2}\phi(x)\right)\sin\left(\frac{\alpha-\beta}{2}\phi(x)\right) & 0\\ \cos\left(\frac{\alpha+\beta}{2}\phi(x)\right)\sin\left(\frac{\alpha-\beta}{2}\phi(x)\right) & -\sin\left(\frac{\alpha+\beta}{2}\phi(x)\right)\sin\left(\frac{\alpha-\beta}{2}\phi(x)\right) & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}N_1(x)\\ N_2(x)\\ N_3(x)\end{pmatrix} \tag{41}$$

$$\begin{pmatrix}\overline{\pi}_1^+(x)\\ \overline{\pi}_2^+(x)\end{pmatrix}=\exp\left(\mathrm{i}\frac{\nu\phi(x)}{2}\right)\begin{pmatrix}\cos\left(\frac{\alpha+\beta}{2}\phi(x)\right) & -\sin\left(\frac{\alpha+\beta}{2}\phi(x)\right)\\ \sin\left(\frac{\alpha+\beta}{2}\phi(x)\right) & \cos\left(\frac{\alpha+\beta}{2}\phi(x)\right)\end{pmatrix}\begin{pmatrix}\pi_1^+(x)\\ \pi_2^+(x)\end{pmatrix} \tag{42}$$

$$\overline{\pi}_3^+(x)=\exp\left(\mathrm{i}\frac{\nu\phi(x)}{2}\right)\left[\cos\left(\frac{\alpha-\beta}{2}\phi(x)\right)\pi_3^+ +\frac{1}{2}\Delta^+(x)\sin\left(\frac{\alpha-\beta}{2}\phi(x)\right)\right] \tag{43}$$

where $\phi(x)=\sum_{i=1}^4\phi_i(x)$, $\theta_{43}=\beta$, $\theta_{12}=\alpha$, $\theta_{13}-\theta_{42}=\nu$, SC order parameter $\Delta^+(x)=-\mathrm{i}c^+(x)\sigma_2 d^+(x)$ and AF order parameter $\vec{N}(x)=\frac{1}{2}(c^+(x)\vec{\sigma}c(x)-d^+(x)\vec{\sigma}d(x))$.

From the above analysis we see that under the $U(1)$ transformation equations (6), (37) still obey Yangian algebra as equation (35) does if θ_{ij} satisfy the condition equations (10), (11) and (38). This indicates that there is a local $U(1)$ gauge invariance in the construction of the current realization for $Y(SO(5))$. It turns out that after the transformation, there are local phase factors in the current realization of $Y(SO(5))$ (shown by equations (41)–(43), but equation (37) still satisfies $Y(SO(5))$ Yangian algebraic relations, i.e. there is a local $U(1)$ gauge invariance in such a current realization of $Y(SO(5))$.

We also find that the transformation equation (6) can be used to diagonalize the massless Thirring model with $SO(5)$ symmetry, that will help to understand the physical meaning of the introduced local $U(1)$ symmetry.

In another words, H' is the Hamiltonian expressed by $\Phi_i(x)$; under the transformation equation (6), it becomes H where two-body interaction appears. So the physical meaning of $U(1)$ transformation in the current realization of $Y(SO(5))$ is connected with two-body interaction in this physical model. Applying the transformation we find the local $U(1)$ gauge invariance in $Y(SO(5))$ explicitly.

We note that there are some non-trivial phase factors in the generators of the Yangian, but they still satisfy the commutation relations of $Y(SO(5))$, i.e. there is a local $U(1)$ gauge invariance in $Y(SO(5))$.

4. Conclusion

Using a local $U(1)$ transformation connecting the four-component fermionic field operator $\psi_i(x)$ with the q-deformed one $\Phi_i(x)$, it is helpful to diagonalize the massless Thirring model with $SO(5)$ symmetry. The Bethe ansatz wavefunction is obtained in a simple manner. It turns out that the current realization of $Y(SO(5))$ is not unique and there exists a local $U(1)$ gauge transformation. This shows the existence of a local $U(1)$ symmetry in the current realization of $Y(SO(5))$. Correspondingly, the transformation leads to the local $U(1)$-gauge invariance for $Y(SO(5))$. The explicit forms of phase factors for $SO(5)$ have been shown.

The authors would like to thank Dr Jing-Ling Chen for helpful discussion. This work is, in part, supported by the NSF of China.

References

[1] Zhang S C 1997 *Science* **275** 1089
[2] Meixner S, Hanke W, Demler E and Zhang S C 1997 *Phys. Rev. Lett.* **79** 4902
[3] Eder R, Hanke W and Zhang S C 1998 *Phys. Rev.* B **57** 13 781
[4] Henly C L 1998 *Phys. Rev. Lett.* **80** 3590
[5] Rabello S, Kohno H, Demler E and Zhang S C 1998 *Phys. Rev. Lett.* **80** 3586
[6] Scalapinp D J, Zhang S C and Hanke W 1998 *Phys. Rev.* B **58** 443
[7] Shelton D G and Sénéchal D 1997 *Preprint* cond-mat/97110251
[8] Komori Y and Wadati M 1996 *J. Phys. Soc. Japan* **65** 722–72
[9] Komori Y and Wadati M 1996 *Phys. Lett.* A **218** 42–8
[10] Wadati M 1988 *Phys. Rev. Lett.* **60** 635
[11] Haldane F D M 1988 Exact Jastrow–Gutzwiller resonating-valence-bond ground state of spin-$\frac{1}{2}$ antiferromagnetic Heisenberg chain with $\frac{1}{r^2}$ *Phys. Rev. Lett.* **60** 635
[12] Ge M-L, Xue K and Cho Y M 1998 Realization of Yangian in quantum mechanics and application *Phys. Lett.* A **249** 358–62
Ge M-L and Xue K $Y(su(3))$ in quantum mechimics *Nankai Math. Inst. Preprint*
[13] Korepin V D, Bogoliubov N M and Izergin A G 1993 *Quantum Inverse Scattering Method and Correlation Functions* (Cambridge: Cambridge University Press)
[14] Haldane F D M 1992 Yangian symmetry of integrable quantum chain with long-range interactions and a new description of states in conformal field theory *Phys. Rev. Lett.* **69** 2021
[15] Zamolodchikow A and Zamolodchikow Al 1979 *Ann. Phys., NY* **120** 25
[16] Faddeev L D 1980 *Sov. Sci. Rev.* C **1** 107
[17] Drinfled V 1985 *Sov. Math. Dokl.* **32** 254
[18] Drinfled V 1985 *Sov. Math. Dokl.* **36** 212